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DECOMPOSITIONS INTO SUBMANIFOLDS THAT YIELD GENERALIZED MANIFOLDS

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The paper sets forth several examples of decompositions of $(n+k)$-manifolds ($k>2$) such that the associated space B is not a generalized manifold. On the other hand, a fundamental technical result, derived from a spectral sequence argument, reveals that B always has cohomological dimension k; the main result attests that if each g in G has trivial Čech homology in dimensions $1, 2, \ldots, k-1$, where $n \geq k \geq 2$, and if B is finite dimensional, then it is a generalized k-manifold. By way of application, when each $g \in G$ has the shape of S^n, it follows that $p: M \to B$ is an approximate fibration.

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Introduction

Let G denote an upper semicontinuous decomposition of an $(n+k)$-manifold M into closed, connected n-manifolds. What can be said about the decomposition space $B = M/G$? Is B an ANR? Under what conditions? If it is, then under what conditions is the decomposition map $p: M \to B$ an approximate fibration? Such questions are the subject of this paper.

Certain forms of these questions have been addressed by Coram and Duvall. After introducing the concept of an approximate fibration [3], they provided several characterizations [4]. In addition, given a decomposition G of S^3 into compacta having the shape of the circle for which S^3/G is known to be S^2, they showed that $p: S^3 \to S^3/G$ is an approximate fibration over the complement of a finite set [5]. More generally, given a decomposition G of an $(n+k)$-manifold M into UV^1 continua such that M/G is a k-manifold, they determine rather minimal movability

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conditions on the map $p: M \to M/G$ implying that p is an approximate fibration over the complement of a locally finite set [7].

When k is 1 or 2 the general problem is fairly well understood. Liem has proved that if G is a usc (meaning, upper semicontinuous) decomposition of an $(n+1)$-manifold into n-spheres, then B is a 1-manifold (possibly with boundary if $n = 1$) and, for $n > 4$, $p: M \to B$ can be approximated by a bundle map, with S^n as the fiber [20]. Daverman has shown that for any usc decomposition G of such an M into (connected) n-manifolds, B is a 1-manifold, possibly with boundary, but $p: M \to B$ need not be an approximate fibration (not even if $\partial B = \emptyset$) [8]. In an earlier paper the authors proved that if G is a usc decomposition of an orientable $(n+2)$-manifold into n-spheres ($n > 0$), then B is a 2-manifold without boundary and $p: M \to B$ is an approximate fibration, unless $n = 1$, in which case p is an approximate fibration over the complement of a locally finite set [11]. Furthermore, the authors have established that for any usc decomposition G of an $(n+2)$-manifold M into orientable, connected n-manifolds, $B = M/G$ is a 2-manifold, possibly with boundary if M itself is nonorientable [12]. The first author refined the last result by removing the assumption that the elements of G be orientable [9].

The paper sets forth several examples of decompositions of $(n+k)$-manifolds ($k > 2$) such that the associated space B is not a generalized manifold. On the other hand, a fundamental technical result, derived from a spectral sequence argument, reveals that B always has cohomological dimension k; the main result attests that if each g in G has trivial Čech homology in dimensions $1, 2, \ldots, k-1$, where $n \geq k \geq 2$, and if B is finite dimensional, then it is a generalized k-manifold. By way of application, when each $g \in G$ has the shape of S^n, it follows that $p: M \to B$ is an approximate fibration.

Assuming that B is finite dimensional, we investigate what can be said when the elements of G satisfy certain more general compatibility conditions concerning homology and, occasionally, homotopy. If the elements of G have isomorphic Čech homology groups in dimensions $1, 2, \ldots, k-2$ and if the appropriate homology carriers are properly aligned in M, then B must be an ANR; if the same situation prevails through dimension k, then B is a generalized k-manifold. These homology carriers will be properly aligned if a local constancy feature holds throughout B, which amounts to the property that each element of G is equipped with a neighborhood V on which there exists a (shape) retraction $r: V \to g$ that induces a degree one map $g' \to g$, for every $g' \in G$ contained in V. Finally, if each $g \in G$ has the shape of some fixed n-manifold N, we give conditions on $\pi_1(N)$ which, when coupled with the local constancy feature above, imply that $p: M \to B$ is an approximate fibration.

1. Computing cohomological dimension

The setting throughout this section is a proper map $f: X \to Y$ between complete, separable metric spaces such that each $f^{-1}(y)$ has the shape of a closed, connected orientable n-manifold. The current focus is to set forth background material necessary for describing the Leray-Grothendieck spectral sequence of a map in a reasonably coherent fashion. Additional details can be extracted from several sources including [16], [25], and [2]. Having this algebraic tool at our disposal, we shall compute that the cohomological dimension $\dim_Z Y \leq \dim_Z X - n$.

Generally, we shall use H^q and H_q to denote Čech cohomology and homology, at times, taking coefficients in a sheaf. Whenever the coefficient group is suppressed, it is the integers.

Associated with a map $f: X \to Y$ is a family of presheaves $\mathscr{H}^q[f]$, $q = 1, 2, \ldots$. The value of the presheaf $\mathscr{H}^q[f]$ on an open subset $U \subset Y$ is the group $H^q(f^{-1}(U); Z)$ and, for each inclusion $i: V \to U$ of open sets, the homomorphism $r_{UV}: \mathscr{H}^q[f](U) \to \mathscr{H}^q[f](V)$ is the induced homomorphism $i^*: (f^{-1}(U); Z) \to H^q(f^{-1}(V); Z)$. For maps f that are proper (i.e., $f^{-1}(K)$ is compact for $K \subset Y$ compact) the stalks of the associated sheaf, which is also denoted by $\mathscr{H}^q[f]$, are the groups $H^q(f^{-1}(x); Z)$.

Theorem (Leray-Grothendieck). *For a map $f: X \to Y$ there is a first quadrant spectral sequence $\{E_r = E_r(f): r \geq 2\}$ such that:*

(1) $E_2^{p,q} = H^p(Y; \mathscr{H}^q[f])$;

(2) $E_\infty^{p,q}$ *is associated to a filtration of* $H^{p+q}(X; Z)$;

(3) *the edge homomorphism is* f^*: $H^p(Y; Z) \approx E_2^{p,0} \to E_\infty^{p,0} \to H^p(X; Z)$.

Remarks. (a) $E_{r+1}^{p,q} = \ker(d_r)/\mathrm{im}(d_r)$ where the differential $d_r^{p,q}: E_r^{p,q} \to E_r^{p+r,q-r+1}$ has bidegree $(r, 1-r)$.

(b) $E_r^{p,q} = E_{r+1}^{p,q} = \cdots = E_\infty^{p,q}$ for $r \geq p+q+1$.

(c) The filtration is $0 \subset J_0 \subset J_1 \subset \cdots \subset J_p = H^p(X; Z)$ where $J_0 = E_\infty^{p,0}$ and $J_i/J_{i-1} = E_\infty^{p-i,i}$, $i = 1, 2, \ldots$.

The *integral cohomological dimension* of a space X is the smallest integer $q \geq 0$ such that $H^{q+1}(A, B; Z) \approx 0$ for every pair of closed subsets $B \subset A$ of X. We denote this measure of dimension by $\dim_Z X$, reserving the notation $\dim X$ to signify the covering dimension. A rather thorough introduction to cohomological dimension can be extracted from [21] and an elementary exposition concentrating on a comparison with covering dimension appears in [26].

Proposition 1.1. *Let X be a complete separable metric space and let $f: X \to Y$ be a proper map such that each set $f^{-1}(y)$ has the shape of a closed, connected orientable n-manifold. Then Y contains a nonempty open set U with* $\dim_Z U \leq \dim_Z X - n$.

Proof. An appeal to [10, Section 2] leads to the conclusion that there is a $y_0 \in Y$ and an open subset U containing y_0 for which there is a shape retraction $r: f^{-1}(U) \to f^{-1}(y_0)$ whose restriction $r|f^{-1}(y) \to f^{-1}(y_0)$ is a shape equivalence for each $y \in U$.

Specify a shape equivalence $e: f^{-1}(y_0) \to N$ to a closed orientable n-manifold and define $h: f^{-1}(U) \to U \times N$ by setting $h(x) = (f(x), e \circ r(x))$. Evidently the restriction induces isomorphisms

$$(h|)^*: \ H^q(\{y\} \times N; Z) \approx H^q(f^{-1}(y); Z), \quad q = 1, 2, \ldots,$$

for each $y \in U$.

Claim. For each closed subset $A \subset U$, the restriction induces isomorphisms $(h|)^*: H^q(A \times N; Z) \to H^q(f^{-1}(A); Z)$, $q = 1, 2, \ldots$.

Prior to proving this, we use it to deduce the inequality $\dim_Z U \leq \dim_Z X - n$. The case of interest is that with $\dim_Z X = m < \infty$. An easy consequence of the claim and the Five Lemma [24, p. 185] is that

$$(h|)^*: \ H^q(A \times N, B \times N; Z) \approx H^q(f^{-1}(A), f^{-1}(B); Z)$$

for each pair of closed subsets $B \subset A$ of U. Since $H^n(N; Z) \approx Z$, were there a closed pair $B \subset A$ in U with $H^{m-n+1}(A, B; Z) \neq 0$, then the Kunneth Formula [24, p. 235] would show that $H^{m+1}(A \times N, B \times N; Z) \neq 0$. Then the isomorphism $(h|)^*$ would detect an impossibility, namely, $H^{m+1}(f^{-1}(A), f^{-1}(B); Z) \neq 0$.

The claim itself is a consequence of the following comparison of the spectral sequence of the restriction $f|: f^{-1}(A) \to A$ with that of the projection map $\pi: A \times N \to A$. Since h induces a 'stalk by stalk' isomorphism $\mathscr{H}^q[\pi] \to \mathscr{H}^q[f|]$, its restriction induces isomorphisms

$$(h|)^*: \ H^p(A; \mathscr{H}^q[\pi]) \approx H^p(A; \mathscr{H}^q[f]), \quad p, q = 0, 1, \ldots .$$

As the restriction $(h|)^*$ induces isomorphisms between E_2 terms of these spectral sequences, their naturality allows us to conclude that

$$(h|)^*: \ H^q(A \times N; Z) \approx H^q(f^{-1}(A); Z), \quad q = 0, 1, \ldots . \quad \square$$

Theorem 1.2. *Let X be a complete separable metric space and let $f: X \to Y$ be a proper map such that each set $f^{-1}(y)$ has the shape of a closed, connected orientable n-manifold. Then* $\dim_Z Y \leq \dim_Z X - n$.

Proof. The case of interest is that with $\dim_Z X = m < \infty$. Since there is a Sum Theorem for cohomological dimension [21], there is a largest open subset $W \subset Y$ for which $\dim_Z W \leq m - n$. Consider the restriction $f|: f^{-1}(Y - W) \to Y - W$. The preceding proposition states that there is a nonempty (provided $Y - W \neq \emptyset$) relatively open subset U of $Y - W$ for which $\dim_Z U \leq \dim_Z f^{-1}(Y - W) - n$ and, hence, $\dim_Z U \leq m - n$. As both W and U are F_σ's, the Sum Theorem detects that $\dim_Z W \cup U \leq m - n$. This is a contradiction unless $W = Y$ for $W \cup U$ would be an open set larger than W. $\square$

Corollary 1.3. *Suppose G is a usc decomposition of an $(n+k)$-manifold M into compacta having the shape of closed, orientable n-manifolds, then* $\dim_Z M/G = k$. *Furthermore, if* $\dim M/G < \infty$, *then* $\dim M/G = k$.

Proof. The preceding theorem ensures that $\dim_Z M/G \leqslant k$, while the result discussed in the next section together with [10, Theorem 2.10] establish that a dense open subset of M/G is a homology k-manifold, revealing that $\dim_Z M/G \geqslant k$. Since the different concepts of dimension are known to coincide for finite dimensional spaces, $\dim M/G = k$ if $\dim M/G < \infty$. □

2. Homology manifolds

The setting throughout this section is an usc decomposition G of an $(n+k)$-manifold M into compacta having the shape of closed orientable n-manifolds. The decomposition space M/G is denoted by B and the induced quotient map by $p: M \to B$. Since the elements of the decompositions considered in this paper are manifolds (up to shape), they are in particular nearly 1-movable and, consequently, the decomposition space B is locally simply connected [13]. There are two possible 'obstructions' that might keep B from being an ANR: the first is that B may not be locally homologically connected (over Z), and the second is that B may not be finite dimensional (though Theorem 1.2 records that B has finite cohomological dimension). We adopt the following terminology. A *generalized k-manifold* is a finite dimensional ANR X that satisfies

$$H_q(X, X-\{x\}; Z) \approx 0 \quad \text{for } q \neq k \text{ and } H_k(X, X-(x); Z) \approx Z.$$

We define *homology k-manifold* by replacing, in the preceding definition, 'finite dimensional ANR X' by 'a homologically locally connected (over Z) space X having finite cohomological dimension'. Since the homology manifolds we encounter are locally simply connected, either singular or Čech homology theory can be used.

The next result is essentially a 'warm-up' for what follows in later sections and is related to results in [10] and [22].

Theorem 2.1. *Suppose M is an $(n+k)$-manifold and suppose G is a usc decomposition of M into compacta having the shape of closed, orientable n-manifolds satisfying: for each $g \in G$, there is a shape retraction $r_g: U \to g$, of a neighborhood U of g, such that the restriction $r|_{g'}: g' \to g$ induces isomorphisms $H^q(g; Z) \approx H^q(g'; Z)$ for all q and all $g' \in G$ with $g' \subset U$.*

Then M/G is a homology k-manifold (and is a generalized k-manifold provided $\dim M/G < \infty$*).*

Proof. Corollary 1.3 states that $\dim_Z M/G \leqslant k$ and the results in [14] imply that M/G is homologically locally connected. Set $B = M/G$ and let $p: M \to B$ denote the quotient map induced by the decomposition. It remains to check that, for $b \in B$, the relative homology groups $H_q(B, B-\{b\}; Z) \approx 0$ for $q \neq k$ and $H_k(B, B-\{b\}; Z) \approx Z$. Since $\dim_Z B \leqslant k$, $H^q(B, B-\{b\}; Z) \approx 0$ for $q > k$ and, hence, $H_q(B, B-\{b\}; Z) \approx 0$ for $q > k$ (see the Universal Coefficient Theorem [24, p. 243]). Since we can 'lift' the decomposition to the oriented double cover in the case that M is not orientable, we assume that M itself is orientable. Since the Mayer–Vietoris Axiom detects that $H_q(B, B-\{b\}) \approx H_q(V, V-\{b\})$ for each neighborhood V of B, it suffices to compute the homology groups $H_q(V, V-\{b\})$ where V is chosen so that there is a shape retraction $r: p^{-1}(V) \to p^{-1}(b)$ whose restriction induces isomorphisms from $H^q(p^{-1}(b))$ to $H^q(p^{-1}(b'))$ for each $b' \in V$. Let $e: p^{-1}(b) \to N$ be a shape equivalence to a closed orientable n-manifold. The spectral sequence analysis in the proof of Proposition 1.1 shows that the map $h: p^{-1}(V) \to V \times N$ defined by setting $h(x) = (p(x), e \circ r(x))$ induces isomorphisms, for $q \geq 0$,

$$(h|)^* = H^q(V \times N, (V-\{b\}) \times N) \approx H^q(p^{-1}(V), p^{-1}(V-\{b\}))$$

and, as a consequence of universal coefficient theorems, induces isomorphisms, for $q \geq 0$,

$$(h|)_*: \ H_q(p^{-1}(V), p^{-1}(V-\{b\})) \approx H_q(V \times N, (V-\{b\}) \times N).$$

Combining this with the duality isomorphism

$$H^j(p^{-1}(b)) \approx H_{n+k-j}(p^{-1}(V), p^{-1}(V-\{b\}))$$

and the Kunneth formula, we obtain

$$H^j(p^{-1}(b)) \approx \left\{ \sum_{0 \leq i \leq n+k-j} H_i(V, V-\{b\}) \otimes H_{n+k-j-i}(N) \right\} + \text{Torsion}.$$

For $j > n$, these groups are trivial (as $H^j(p^{-1}(b)) \approx 0$). Using the fact that $H_0(N) \approx Z$, we conclude that $H_q(V, V-\{b\}) \approx 0$ for $q < k$. Knowing that these groups are trivial, since $H^n(p^{-1}(b)) \approx Z$, we use this same isomorphism to conclude that $H_k(V, V-\{b\}) \approx Z$. □

3. Decompositions and generalized manifolds

The primary result of this section is the theorem given below. A substantial application, concerning decompositions into spheres, is treated in Section 4, and a further improvement is developed in Section 5.

Theorem 3.1. *Suppose M is an $(n+k)$-manifold, $n \geq k \geq 3$, and suppose G is a usc decomposition of M into compacta having the shape of closed, orientable n-manifolds satisfying:*

(1) *the reduced homology $H_i(g; Z) \approx 0$ for all $g \in G$ and $i \in \{0, 1, \ldots, k-1\}$, and*

(2) $\dim M/G < \infty$.

Then M/G is a generalized manifold.

Proof. Localizing about $g \in G$, we can assume without loss of generality that M is orientable; each $g \in G$ has a neighborhood W_g such that the image of $H_1(W_g) \to H_1(M)$ is trivial, which implies the orientability of W_g. (We issue a reminder that suppressed coefficients are always the integers.)

By (2) above and Corollary 1.3, $\dim[B = M/G] = k$. It follows trivially from (1) that, in the terminology of Dydak and Segal [14], the decomposition map $p: M \to B$ is homology $(k-1)$-stable. By Corollary 4.9 of [14], B is LC^k. Hence, B is a finite dimensional ANR.

It remains to show that $H_j(B, B-\{b\})$ is isomorphic to Z when $j = k$ and is trivial otherwise. The latter is straightforward. According to the Vietoris-Begle Mapping Theorem [1] and (1) above, p induces homology isomorphisms

$$p_*\colon\ H_i(M, M-p^{-1}(b)) \to H_i(B, B-\{b\})$$

for $i \in \{0, 1, \ldots, k-1\}$. Duality [24] then reveals that, in this range,

$$H_i(B, B-\{b\}) \approx H_i(M, M-p^{-1}(b)) \approx H^{n+k-i}(p^{-1}(b)) \approx 0.$$

If each $g \in G$ were known to satisfy $H_k(g) \approx 0$, as well as (1), then the same argument could be used to obtain

$$H_k(B, B-\{b\}) \approx H_k(M, M-p^{-1}(b)) \approx H^n(p^{-1}(b)) \approx Z.$$

What follows is designed to circumvent this extra hypothesis. [The Vietoris-Begle Mapping Theorem also states that the homomorphism $p_*: H_k(M, M-p^{-1}(b)) \to H_k(B, B-\{b\})$ is a surjection; and, hence, there are limited possibilities for $H_k(B, B-\{b\})$.] Conceptually there are two distinct parts:

(I) the proof that the set X of points at which B fails to be a generalized k-manifold is (at worst) countable and closed in B, and

(II) the proof that X contains no isolated points.

The next lemma gives algebraic data used in (I); its proof is also used in (II).

Lemma 3.2. *Suppose that, for any two neighborhoods U^* and V^* of $g_0 \in G$, with $V^* \subset U^*$,*

$$\operatorname{im}\{H_k(V^* - g_0) \to H_k(U^*)\} \supset \operatorname{im}\{H_k(g_0) \to H_k(U^*)\}.$$

Then $H_k(B, B - p(g_0)) \approx Z$.

Proof. Given a neighborhood W of $p(g_0)$ in B, find a smaller neighborhood U of $p(g_0)$ that is contractible in W and find yet a smaller neighborhood V of $p(g_0)$ that is contractible in U. Set $U^* = p^{-1}(U)$ and $V^* = p^{-1}(V)$, and restrict U and V, if necessary, so that

$$\operatorname{im}\{H_k(V^*) \to H_k(U^*)\} = \operatorname{im}\{H_k(g_\theta) \to H_k(U^*)\},$$

$$\operatorname{im}\{H_k(U^*) \to H_k(W^*)\} = \operatorname{im}\{H_k(g_0) \to H_k(W^*)\}, \quad \text{and}$$

$$\operatorname{im}\{H_{k-1}(V^*) \to H_{k-1}(U^*)\} \approx 0.$$

From the commutative diagram,

$$\begin{array}{ccccccc}
H_k(V) & \to & H_k(V, V-p(g_0)) & \to & H_{k-1}(V-p(g_0)) & \to & H_{k-1}(V) \\
\downarrow 0 & & \downarrow \approx & & \downarrow \gamma_{k-1} & & \downarrow 0 \\
H_k(U) & \to & H_k(U, U-p(g_0)) & \to & H_{k-1}(U-p(g_0)) & \to & H_{k-1}(U) \\
\downarrow & & \downarrow \approx & & \downarrow & & \downarrow 0 \\
H_k(W) & \to & H_k(W, W-p(g_0)) & \to & H_{k-1}(W-p(g_0)) & \to & H_{k-1}(W)
\end{array}$$

we find that

$$H_k(B, B-p(g_0)) \approx H_k(U, U-p(g_0)) \approx \text{im } \gamma_{k-1}.$$

Once again the Vietoris–Begle Theorem [1] implies that the homomorphisms shown vertically in the diagram below

$$\begin{array}{ccc}
H_{k-1}(V^*-g_0) & \xrightarrow{\alpha_{k-1}} & H_{k-1}(U^*-g_0) \\
\downarrow (p|)_* & & \downarrow (p|)_* \\
H_{k-1}(V-p(g_0)) & \xrightarrow{\gamma_{k-1}} & H_{k-1}(U-p(g_0))
\end{array}$$

are isomorphisms; thus, im $\gamma_{k-1} \approx$ im α_{k-1}. Consider the homology ladder

$$\begin{array}{ccccccccc}
H_k(V^*-g_0) & \to & H_k(V^*) & \to & H_k(V^*, V^*-g_0) & \to & H_{k-1}(V^*-g_0) & \to & H_{k-1}(V^*) \\
\downarrow \alpha_k & & \downarrow \beta_k & & \downarrow \approx & & \downarrow \alpha_{k-1} & & \downarrow 0 \\
H_k(U^*-g_0) & \to & H_k(U^*) & \to & H_k(U^*, U^*-g_0) & \to & H_{k-1}(U^*-g_0) & \to & H_{k-1}(U^*).
\end{array}$$

The choice of V together with the hypotheses of the lemma insure that im $\alpha_k \to$ im β_k is surjective and it follows that $H_k(V^*, V^*-g_0) \to H_{k-1}(V^*-g_0)$ is injective. The same argument with $U^* \subset W^*$ shows that $H_k(U^*, U^*-g_0) \to H_{k-1}(U^*-g_0)$ is injective. The above homology ladder also reveals that $H_k(U^*, U^*-g_0) \to$ im α_{k-1} is surjective. Thus,

$$\text{im } \alpha_{k-1} \approx H_k(U^*, U^*-g_0) \approx H^n(g_0) \approx Z.$$

Consequently,

$$H_k(B, B-p(g_0)) \approx \text{im } \gamma_{k-1} \approx \text{im } \alpha_{k-1} \approx Z. \qquad \square$$

Now let gm(B) denote the set of points $b \in B$ for which $H^k(B, B-\{b\}) \approx Z$.

Lemma 3.3. $B-\text{gm}(B)$ *has no isolated points.*

Proof. Suppose to the contrary that b is an isolated point of $B-\text{gm}(B)$, so it has a neighborhood W such that $W \cap (B-\text{gm}(B)) = \{b\}$. As in the proof of Lemma 3.2, name connected open sets U and V in W, with $b \in V \subset U \neq B$, such that

$$H_k(B, B-\{b\}) \approx H_k(U, U-\{b\}) \approx \text{im}\{\gamma_{k-1} : H_{k-1}(V-\{b\}) \to H_{k-1}(U-\{b\})\}.$$

It will suffice to establish that $H_k(B, B-\{b\})$ is infinite, for we have previously observed the consequence of the Vietoris–Begle Theorem that

$$p_*: H_k(M, M-p^{-1}(b)) \approx Z \to H_k(B, B-\{b\})$$

is surjective.

Since $k\geq 3$, $H_i(U, U-\{b\})\approx 0\approx H_i(W, W-\{b\})$ for $i=1,2$, and hence, $H_1(U-\{b\})\to H_1(W-\{b\})$ is trivial. Then the generalized k-manifolds $U-\{b\}$ and $V-\{b\}$ are orientable. By duality,

$$H_{k-1}(U-\{b\})\approx H^1_c(U-\{b\}) \text{ and } H_{k-1}(V-\{b\})\approx H^1_c(V-\{b\}).$$

Computing with rational (Q) coefficients, we inspect the initial part of the cohomology sequence for the ends of $V-\{b\}$:

$$0\to H^0(V-\{b\};\, Q)\to H^0_e(V-\{b\};\, Q)\xrightarrow{\varphi} H^1_c(V-\{b\};\, Q).$$

Since $H^0_e(V-\{b\};\, Q)$ is the vector space over Q with basis equal to the set of ends in $V-\{b\}$ [15, Theorem 1], its dimension is at least 2 as a vector space over Q. Moreover, $H^0(V-\{b\};\, Q)\approx Q$, so im φ has dimension at least 1 as a vector space over Q. It follows from Universal Coefficient Theorems that $H^1_c(V-\{b\};\, Z)$ has an element of infinite order.

For simplicity, write $V-\{b\}$ as V_0 and $U-\{b\}$ as U_0. Then V_0 contains a compact set X for which the image of

$$H_{k-1}(X)\to H_{k-1}(V_0)\approx H^1_c(V_0)$$

is infinite. By duality, $\operatorname{im}\{j: H^1_c(V_0, V_0-X)\to H^1_c(V_0)\}$ is infinite. Then inspection of the ladder

$$\begin{array}{ccccc}
0\approx H^0_c(V_0-X) & \to & H^1_c(V_0, V_0-X) & \xrightarrow{j} & H^1_c(V_0)\\
\downarrow & & \downarrow\approx & & \downarrow\kappa\\
0\approx H^0_c(U_0-X) & \to & H^1_c(U_0, U_0-X) & \to & H^1_c(U_0)
\end{array}$$

reveals that im κ is also infinite, and duality does the same for im γ_{k-1}. □

Continuing with (I), we retrace part of the path pursued by Coram and Duvall [7], in their treatment of the n-winding functions. (Caution: here n will be the shape dimension of $g\in G$, not the homology dimension.) For simplicity we will assume each $g\in G$ is an ANR, so that there exists an actual retraction $r_g: p^{-1}(U_g)\to g$ (rather that just a shape retraction) defined over some neighborhood U_g of g, and there exists a smaller neighborhood V_g such that the restriction $r_g|: p^{-1}(V_g)\to g$ and the inclusion $p^{-1}(V_g)\to p^{-1}(U_g)$ are homotopic as maps into $p^{-1}(U_g)$. Then the n-winding function $\alpha_{p(g)}: V_g\to Z$ is defined as

$$\alpha_{p(g)}(c) = \text{absolute degree of } (r_g)_*: H_n(p^{-1}(c))\to H_n(g).$$

Let K denote the set of all $b\in B$ such that, for each neighborhood W of b, $0\in\alpha_b(W)$. By [7, Lemma 3.1], K is nowhere dense in B, and by [7, Lemma 3.4], the set

$$C=\{b\in B-K: \alpha_b \text{ is continuous at } b\}$$

is open and dense in $B-K$. Given $g_0\in G$ for which $p(g_0)\in C$, we see that arbitrarily close to $p(g_0)$ are points c such that

$$(r_g)_*:\ H_n(p^{-1}(c))\to H_n(g_0)$$

is degree one. By Lemma 2.1 of [27], r_g induces epimorphisms $H_i(p^{-1}(c)) \to H_i(g_0)$ in all dimensions. From this one can see directly that the hypothesis of Lemma 3.2 holds at g_0, which implies $C \subset \text{gm}(B)$.

Define D as $(B-K)-C$. Clearly D is closed in $B-K$, so D is a Baire space. As a result, D has no isolated points, for otherwise Lemma 3.3 would be violated. This means that each $d \in D$ is a limit point of $D-\{d\}$. Consequently, the argument of the preceding paragraph establishes that $D \subset \text{gm}(B)$.

Now it follows in similar fashion that $K \subset \text{gm}(B)$. Hence, $B = \text{gm}(B)$, and the proof of Theorem 3.1 is complete. □

With its completion comes a substantial redirection of our aims. Having given sufficient (and far from necessary!) conditions under which M/G is a generalized manifold, we intend to explore in the second part of this section some construction techniques leading to decomposition spaces M/G that are not quite so nice.

First of all, we display the most elementary example.

Example 3.1. A decomposition space that is not a generalized k-manifold, $k \geqslant 3$. Let $n = k-1$. Choose any closed n-manifold N for which $\pi_1(N) \neq \{1\}$. Set $M_A = N \times E^k$, and define G_A as the decomposition of the $(2n+1)$-manifold M_A into $N \times \{0\}$ and the n-spheres $\{z\} \times rS^{k-1}$, $z \in N$ and $r > 0$. Then M_A/G_A is homeomorphic to the (open) cone on N, which fails to be a generalized manifold at the cone point.

Example 3.1 also arises as a special case of the more general spinning construction, which we will exploit several times in turning out later examples.

Let W denote an $(n+1)$-manifold with (nonempty) boundary and $s \geqslant 0$ and integer. The *s-spin of W* (about its boundary) written $\text{Sp}^s\, W$, is defined to be the $(n+s+1)$-manifold $(W \times S^s)/R$, where R represents the decomposition into singletons and the spheres $\{w\} \times S^s$, $w \in \partial W$. One can easily verify that $\text{Sp}^s\, W$ is homeomorphic to the manifold $(W \times S^s) \bigcup_\varphi (\partial w \times B^{s+1})$, the identification space determined by the natural homeomorphism φ of $\partial(W \times S^s)$ to $\partial(\partial W \times B^{s+1}) = \partial W \times \partial B^{s+1}$.

Now we spin an arbitrary usc decomposition G_W of W. By the *s-spin of* G_W, written $\text{Sp}^s\,(G_W)$, we mean the usc decomposition of $\text{Sp}^s\, W$ given as $\{p_R(g \times S^s)\colon g \in G_W\}$, where $p_R\colon W \times S^s \to \text{Sp}^s\, W = (W \times S^s)/R$ is the decomposition map. For those $g \in G$ contained in ∂W, $p_R(g \times S^s)$ is homeomorphic to g, while for those in $\text{Int}\, W$, $p_R(g \times S^s)$ is homeomorphic to $g \times S^s$. Furthermore, $\text{Sp}^s\, W/\text{Sp}^s(G_W)$ is naturally equivalent to W/G_W.

Proposition 3.4. *For any compact $(n+1)$-manifold W with boundary, there exists a usc decomposition G of some closed $(2n+1)$-manifold M into closed S-manifolds such that M/G is homeomorphic to W/G_W, where G_W denotes the decomposition of W into singletons and the components of ∂W.*

Proof. Here $M = \mathrm{Sp}^n W$ and $G = \mathrm{Sp}^n(G_W)$. $\square$

Proposition 3.4 makes it plain that a variety of non-generalized manifolds, like suspensions of most closed manifolds, occur as manifold decomposition spaces.

The spinning construction serves as an efficient device for producing a class of examples larger than those described by Proposition 3.4.

Proposition 3.5. *Let G_N denote a usc decomposition of the n-manifold N into s-manifolds. Then there exists a usc decomposition G of $M = N \times E^{s+1}$ into closed s-manifolds such that M/G is equivalent to the open mapping cylinder of the decomposition map $\pi: N \to N/G_N$.*

Proof. Set $W = N \times [0, 1)$ and let G_W denote the decomposition of W into singletons and $\{g \times \{0\}: g \in G_N\}$. Then W/G_W is topologically the open mapping cylinder of π. Take s-spins of W and G_W, and apply Proposition 3.4. $\square$

The following result identifies some decompositions for which the mapping cylinder of the decomposition map fails to be a generalized manifold.

Proposition 3.6. *Suppose G_N is a usc decomposition of an n-manifold N into closed s-manifolds and suppose there exists $g_0 \in G_N$ such that $H_m(g_0: Z)$ is nontrivial, for some $m \in \{1, 2, \ldots, s-1\}$. Then the open mapping cylinder Y of $\pi: N \to N/G_N$ is not a generalized manifold.*

Proof. Suppose the contrary. Of necessity Y then is a generalized $(n+1)$-manifold. It follows from Corollary 1.3 that the image X, corresponding to N/G_N, of $N \times \{0\}$ is $(n-s)$-dimensional.

Choose an open subset U of N such that $H_m(g_0) \to H_m(U)$ is one to one. We form an open set U^* in Y as the image of $U \times [0, 1)$. By duality (which holds locally in Y), for $j = m$ and $m+1$

$$H_j(U^*, U^* - X) \approx H_c^{n+1-j}(X) \approx 0.$$

Hence, $H_m(U^* - X) \to H_m(U^*)$ is an isomorphism. Equating $U^* - X$ with $U \times (0, 1)$, we see that some nontrivial element of

$$\mathrm{im}\{H_m(g_0) \to H_m(U \times \{\tfrac{1}{2}\}) \to H_m(U \times (0, 1)\}$$

is homotopic in U^* to a constant (trivial) cycle, which is impossible. $\square$

By way of summary we set forth an explicit example.

Example 3.2. The (open) mapping cylinder of the decomposition map $\pi: M_A \to M_A/G_A$ of Example 3.1 is the decomposition space associated with some decomposition of a $(3n+2)$-manifold into closed n-manifolds, but it is not a generalized manifold.

With every one of the examples (and of the techniques) giving a non-generalized manifolds laid out in this section, the decomposition space includes points b at which the n-winding function is degenerate (that is, $0 \in \alpha_b(W)$ for each neighborhood W of b). We shall conclude this section by exhibiting that degeneracy is not essential. Other aspects of the role played by the n-winding functions will be investigated more closely in Section 5.

Let Γ denote a finite cyclic group, with generator λ, acting semifreely on some connected k-manifold N, and let F $(\neq\emptyset, \neq N)$ denote the points left fixed by λ. Consider the mapping torus M_λ of λ; that is, M_λ is the $(k+1)$-manifold resulting from $N\times[-1, 1]$ after identifying each $(y, 1)$ with $(\lambda(y), -1)$, $x\in N$. Consider also the decomposition G_λ of M_λ into the circles corresponding to the images of $\bigcup\{\lambda^i(y)\times[-1, 1]: 1\leq i\leq|\Gamma|\}$. Then the following is obvious.

Proposition 3.7. *The orbit space N/Γ is homeomorphic to M_λ/G_λ and the* 1*-winding functions α_b $(b\in M_\lambda/G_\lambda)$ are nondegenerate.*

It is worth noting that, given $b\in M_\lambda/G_\lambda$ corresponding to the image of the circle determined by $\{y\}\times[-1, 1]$, where $y\in F$, one can find points b' arbitrarily close to b such that $\alpha_b(b')$ equals the period of λ.

Example 3.3. For $k\geq 3$, a usc decomposition G_C of the nonorientable E^k-bundle M_C over S^1 into circles such that M_C/G_C fails to be a generalized manifold and the 1-winding functions on M_C/G_C are nondegenerate. The group Γ is determined by the involution λ of E^k sending $y\in E^k$ to $-y$. Under the foregoing construction, $E^k/\Gamma\approx M_C/G_C$ coincides with the open cone on P^{k-1} (real projective $(k-1)$-space).

A final (straightforward) example based on this technique certifies that orientability of the source manifold can also be attained.

Example 3.4. An usc decomposition G_D of $M_D=S^1\times E^4$ into circles such that M_D/G_D fails to be a generalized manifold and the local 1-winding functions are nondegenerate. For $m\in\{3, 4, 5, \ldots\}$ and $q\in\{1, \ldots, m-1\}$, it should be transparent how to define Γ so that $E^4/\Gamma\approx M_D/G_D$ coincides with the the open cone over the Lens space $L(m, q)$.

4. Decompositions into spheres

Quick application of Theorem 3.1 occurs when G is a decomposition involving compacta with the shape of S^n.

Theorem 4.1. *Suppose M is an $(n+k)$-manifold, $n \geq k \geq 2$, and G is a usc decomposition of M into compacta having the shape of S^n such that* $\dim M/G < \infty$. *Then M/G is a generalized k-manifold and the decomposition map $p: M \to M/G$ is an approximate fibration.*

Remark. When $k=2$ this is derived in [11, Theorem 5.2]. If we invoke only the weaker Theorem 4.4 of [12], which attests that M/G is a 2-manifold, we then can develop a unified explanation, for all $k \geq 2$, of why p is an approximate fibration.

Proof of Theorem 4.1. According to Theorem 3.1 here (or [12, Theorem 4.4] in case that $k=2$), $B = M/G$ is a generalized k-manifold.

Fix $g = p^{-1}(b) \in G$. Since $p_*: H_k(M, M-g) \to H_k(B, B-\{b\})$ then is an epimorphism [1] of one infinite cyclic group ($\approx H^n(g)$) onto another, p_* is an isomorphism. The commutative diagram, where U is a connected open neighborhood of b having compact closure, and where $c \in U$,

$$\begin{array}{ccccc}
H^n(g) \approx H_k(M, M-g) & \xrightarrow{p_* \approx} & H_k(B, B-\{b\}) & \approx & H^0(\{b\}) \\
\uparrow & & \uparrow & & \uparrow\approx \\
H_k(M, M-\mathrm{cl}(U)) & \xrightarrow{p_* \approx} & H_k(B, B-\mathrm{cl}(U)) & \approx & H^0(\mathrm{cl}(U)) \\
\downarrow & & \downarrow\approx & & \downarrow\approx \\
H^n(p^{-1}(c)) \approx H_k(M, M-p^{-1}(c)) & \xrightarrow{p_* \approx} & H_k(B, B-\{c\}) & \approx & H^0(\{c\})
\end{array}$$

can be put to work to demonstrate that the winding function α_b^* defined on the nth cohomology is constantly 1 near b, implying the same for the winding function on nth homology. In other words, $p: M \to B$ is H_n-movable. It follows from [6, Theorems A and B] that p is an approximate fibration. □

Corollary 4.2. *For $n \geq k \geq 1$ there exists no usc decomposition of either S^{n+k} or E^{n+k} into n-spheres.*

Proof. The Jordan–Brouwer Separation Theorem readily disposes of the case $k=1$. Specifically, this is an application of [8, Corollary 3.7] when S^{n+1} is the source.

For $k \geq 2$ suppose there exists a usc decomposition of, first S^{n+k}, with decomposition map $p: S^{n+k} \to B$. Under the assumption that $\dim B < \infty$, Theorem 4.1 and Corollary 3.5 of [3] give an exact sequence

$$0 = \pi_k(S^{n+k}) \to \pi_k(B) \to \pi_{k-1}(S^n) = 0.$$

But this is impossible, because B is a closed generalized k-manifold for which $\pi_j(B)$ is trivial, $j \in \{1, \ldots, k-1\}$, by the Vietoris Mapping Theorem in homotopy [23], so $\pi_k(B) \approx H_k(B) \neq \{0\}$ [24, p. 398].

Second, suppose there exists a usc decomposition of E^{n+k}, with decomposition map $p: E^{n+k} \to B$. Still under the assumption that $\dim B < \infty$, there is an exact sequence, as before, where

$$\pi_{n+1}(B) \to \pi_n(S^n) \to \pi_n(E^{n+k}) = 0.$$

This is again impossible, for now B is simply connected and homologically trivial in all positive dimensions, yielding $\pi_{n+1}(B)=0$.

Without the assumption that $\dim B<\infty$, virtually the same argument can be applied, essentially because the exact sequence from [3, Corollary 3.5] still holds in the context at hand. The proof of Theorem 4.1 indicates that the map p from the source to B is homology stable and, therefore, [14, Lemma 2.6] is stable (equivalently, completely movable). As a result, p has the approximate homotopy lifting property for all finite-dimensional separable metric spaces (cf. the discussion of [6, Section 2]). Thus, the conclusion of Theorem 2.4 in [3] is valid for the given map p, and this result is what upholds the aforementioned exact sequence. $\square$

Theorem 4.1 is sharp, in that there are values $k>n$ for which there exist decompositions of some M^{n+k} into n-spheres such that the decomposition map is not an approximate fibration. One source is the map $f: S^{2n+1}\to S^{n+1}$, whose point inverses are n-spheres, described by Lacher [18].

The next example confirms the need for some restriction on n and k in order to attain the other conclusion of Theorem 4.1 that M/G is a generalized manifold.

Example 4.1. A usc decomposition G of S^{n+k} ($k=2n+2$) into n-spheres such that S^{n+k}/G is a finite dimensional ANR but not a generalized manifold. Let G_S denote the decomposition of S^{2n+1} induced by the Lacher map [18] $f: S^{2n+1}\to S^{n+1}$. Explicitly, G_S consists of an exceptional element g_e, which is standardly embedded in S^{2n+1}, and other elements corresponding to $S^n\times\{z\}$ in $S^n\times E^{n+1}=S^{2n+1}-g_e$. Extend G_S from $S^{2n+1}=\partial B^{2n+2}$ to a decomposition G_B of B^{2n+2}, where the elements of G_B-G_S are just singletons. Then $S^{3n+2}\approx \mathrm{Sp}^n(B^{2n+2})$. Define G as $\mathrm{Sp}^n(G_B)$. It follows that $S^{3n+2}/G\approx B^{2n+2}/G_B$, and the images of $(B^{2n+2}-g_e)$ in the latter is equivalent to $S^{n+1}\times E^{n+1}$. As a result, S^{3n+2}/G is topologically the one-point compactification of $S^{n+1}\times E^{n+1}$.

Iterating the construction of Example 4.1, one can produce a usc decomposition of $S^{(j+1)n+j}$ into n-spheres for $j\in\{1,2,\ldots\}$. In conjunction with Corollary 4.2, this sheds light on the unsatisfactorily resolved:

Question. For which integers $m>n>0$ does there exist a usc decomposition of S^m into n-spheres?

5. Decompositions with locally constant n-winding functions

In this part we analyze decompositions G into orientable n-manifolds (up to shape) in which two additional properties prevail:

(1) the n-winding functions on $B=M/G$ are locally constant, and

(2) all decomposition elements have the same Čech homology.

First we show that, with the usual extra condition $\dim M/G<\infty$, B is an ANR. Second, when all $g\in G$ are simply connected, up to shape, we verify that $p: M\to B$

is an approximate fibration. Then we provide an example displaying the independence of (1) and (2). Finally, we focus on the situation in which all $g \in G$ have the same shape, with limitations as found in Section 3 on n and k, and derive further results about $p: M \to B$ being an approximate fibration.

Lemma 5.1. *Let M be an $(n+k)$-manifold and G a usc decomposition of M into continua having the shape of n-manifolds. Suppose* (i) $\dim[B = M/G] < \infty$ *and* (ii) *the decomposition map $p: M \to B$ is homology $(k-1)$-stable. Then B is an ANR.*

Proof. By Corollary 1.3, $\dim B \leq k$. According to Corollary 4.10 of [14], B is LC^k, and thus B is an ANR. □

Proposition 5.2. *Suppose M is an $(n+k)$-manifold and G is a usc decomposition of M into continua having the shape of orientable n-manifolds. Suppose*

(0) $\dim[B = M/G] < \infty$,

(1) *the n-winding functions α_b defined on B are locally constant, and*

(2) *for any two $g_1, g_2 \in G$ and $i \in \{1, 2, \ldots, k-2\}$, $H_i(g_1) \approx H_i(g_2)$.*

Then B is an ANR.

Proof. Fix $g \in G$. Since g is an FANR, there exists a neighborhood V_g of g in M on which is defined a shape retraction $r: V_g \to g$, shape homotopic to the inclusion $V_g \to M$ (relative to g). Then (1) assures that, for all $g' \in G$ sufficiently close to g, $(r|)_*: H_n(g') \to H_n(g)$ is an isomorphism. It follows from [27] that $(r|)_*: H_i(g') \to H_i(g)$ is an epimorphism for all i, and (2) assures that it is an isomorphism for $i < k-1$. Consequently, $p: M \to B$ is homology $(k-1)$-stable, so Lemma 5.1 applies. □

With slightly stronger hypotheses on the homology stability of p, the conclusion about B can be strengthened.

Lemma 5.3. *If in Lemma 5.1 M is orientable and the decomposition map $p: M \to B$ is homology k-stable, then B is a generalized k-manifold.*

Proof. By Lemma 5.1 and Corollary 1.3, B is a k-dimensional ANR. Fix $b \in B$, and find a small neighborhood V of b such that $p^{-1}(V)$ shape deformation retracts to $p^{-1}(b)$ in M. The spectral sequence argument of Proposition 1.1 gives

$$H_i(p^{-1}(V), p^{-1}(V - \{b\})) \approx H_i(V \times g, (V - \{b\}) \times g),$$

where $g = p^{-1}(b)$, for $i \in \{1, 2, \ldots, k\}$. By duality in M,

$$H_i(p^{-1}(V), p^{-1}(V - \{b\})) \approx H^{n+k-i}(g),$$

which implies these groups are trivial when $i < k$ and are isomorphic to Z when $i = k$. A simple application of the Kunneth formula reveals $H_i(V, V - \{b\}) \approx 0$ for $i < k$ and $H_k(V, V - \{b\}) \approx Z$. Hence, B is a generalized k-manifold. □

Theorem 5.4. *Suppose M is an orientable $(n+k)$-manifold and G is a usc decomposition of M into continua having the shape of orientable n-manifolds. Suppose*
(0) $\dim[B = M/G] < \infty$,
(1) *the n-winding functions α_b on B are locally constant, and*
(2) *for any two $g_1, g_2 \in G$ and any $i \in \{1, 2, \ldots, k-1\}$, $H_i(g_1) \approx H_i(g_2)$.*
Then B is a generalized k-manifold.

Proof. See the argument given for Proposition 5.2. □

Remark. Examples 3.3 and 4.1 expose the necessity of hypothesis (1) in Theorem 5.4.

The theorem stated below can be derived without much extra effort.

Theorem 5.5. *Suppose G is an usc decomposition of the $(n+k)$-manifold M into compacta having the shape of simply connected n-manifolds, all with isomorphic Čech homology groups, and suppose the local n-winding functions α_b on $B = M/G$ are locally constant. Then the decomposition map $p: M \to B$ is stable (=completely movable).*

Proof. The argument for Proposition 5.2 shows that p is homology stable. Since each $g \in G$ has the shape of a simply connected manifold, the desired conclusion results from [14, Lemma 2.6]. □

Corollary 5.6. *Under the hypothesis of Theorem* 5.5, *where M is assumed to be connected, all elements of G have the same shape.*

Proof. As outlined near the end of the proof for Corollary 4.2, $p: M \to B$ has the approximate homotopy lifting property for finite dimensional separable metric spaces (i.e., for all $g \in G$). □

Corollary 5.7. *In addition to the hypothesis of Theorem* 5.5, *assume* $\dim[B = M/G] < \infty$. *Then B is a generalized k-manifold and $p: M \to B$ is an approximate fibration.*

Proof. See [6, Theorem B] and [10, Theorem 3.1]. □

One naturally might inquire whether it is necessary to hypothesize in the preceding results both the constancy of the n-winding functions and the homological equivalence of the decomposition elements. Example 3 of [8] displays a decomposition G of some M^{n+1} into n-manifolds for which the n-winding functions α_b on M/G are locally constant but the elements are not all homotopy equivalent. With such codimension one decompositions (into orientable n-manifolds, up to shape), and under the assumption of n-winding constancy, the decomposition elements must be homologically equivalent [8], so their differences are measured in their fundamental groups.

Example 5.1. A usc decomposition of an orientable (connected) $(n+k)$-manifold $(n \geqslant k \geqslant 2)$ into orientable n-manifolds such that the decomposition elements are not pairwise homologically equivalent but the local n-winding functions on the decomposition space are locally constant. These examples are described in Section 5 of [12].

Example 5.2. A usc decomposition of a connected orientable $(n+k)$-manifold $(n \geqslant k \geqslant 2, n \geqslant 3)$ into simply connected n-manifolds such that the decomposition elements are not all homologically equivalent but the local n-winding functions on the decomposition space are locally constant. The only difference between this and Example 5.1 is found at the start; here we begin with a knotted $(n-1)$-sphere K in S^{n+1} with $\pi_1(S^{n+1}-K) \approx Z$ but with $\pi_k(S^{n+1}-K)$ non-trivial for some $k>1$, and whose closed complement fibers over S^1 as before. Such examples are given by Stallings (cf. [17]) and Levine [19].

Example 5.2 illustrates the need for the requirement in Theorem 5.5 that all $g \in G$ have the same homology groups.

Our next goal is to give two extensions of Corollary 5.7. Before doing that, we need a technical result.

Lemma 5.8. *Suppose G is an usc decomposition of a connected $(n+k)$-manifold M into compacta having the shape of orientable n-manifolds. In addition, suppose*:

(i) *the n-winding functions α_b defined on M/G are locally constant, and*

(ii) *each $g \in G$ has a neighborhood W_g such that every inclusion induced $\pi_1(g') \to \pi_1(W_g)$ is one to one ($g' \in G$, $g' \subset W_g$), where $\pi_1(g')$ is the 'shape group' of g' should g' have only the shape of a manifold.*

Then there exists another neighborhood V_g of g in W_g such that for $g' \in G$ with $g' \subset V_{g'}$ the inclusion induced $\pi_1(g') \to \pi(W_g)$ is an isomorphism onto $\operatorname{im}\{\pi_1(g) \to \pi(W_g)\}$.

Proof. For $g \in G$ and W_g as in (ii), there exist a neighborhood V_g of g in W_g and a shape retraction $r_g : V_g \to g$ that is homotopic in W_g to the inclusion. Furthermore, V_g can be chosen small enough that the induced $(r_g)_* : H_n(g') \to H_n(g)$ is an isomorphism for all $g' \in G$ in V_g. Consequently, r_g induces epimorphisms $\pi_1(g') \to \pi_1(g)$ for such g'. Taken with (ii), this gives the lemma. □

Theorem 5.9. *Suppose G is a usc decomposition of an $(n+k)$-manifold M into compacta, all having the shape of some fixed orientable n-manifold N such that the integral group ring $Z[\pi_1(N)]$ is Noetherian. In addition, suppose*:

(i) *the n-winding functions α_b defined on $B = M/G$ are locally constant,*

(ii) *each $g \in G$ has a neighborhood W_g such that every inclusion induced $\pi_1(g') \to \pi_1(W_g)$ is one to one ($g' \in G$, $g' \subset W_g$), and*

(iii) $\dim B < \infty$.

Then B is a generalized k-manifold and the decomposition map $p: M \to B$ is an approximate fibration.

Proof. The strategy here is to show that the decomposition map $p: M \to B$ is (homotopically) stable. As in the proof of Theorem 3.1, B is an ANR, so the desired conclusion will result from [4, Corollary 3.4].

Focus on one $g \in G$. Let W_g be a neighborhood of g as in (ii). Apply Lemma 5.6 to obtain V_g, $g \subset V_g \subset W_g$, such that

$$\operatorname{im}\{\pi_1(g') \to \pi_1(W_g)\} = \operatorname{im}\{\pi_1(g) \to \pi_1(W_g)\} \tag{$*$}$$

for all $g' \in G$ in V_g.

Let $\lambda : W^* \to W_g$ denote the covering determined by $\operatorname{im}\{\pi_1(g) \to \pi_1(W_g)\}$. Consider $g' \in G$ where $g' \subset V_g$. Both $\lambda^{-1}(g')$ and $\lambda^{-1}(g)$ are connected, due to (ii) and ($*$). The shape map $r_g : g' \to g$ behaves like some map $f: N \to N$ of degree one, for which f_* is an isomorphism on the fundamental group level. We can assume f is of geometric degree one, meaning that over some n-cell $B \subset N$, $f|_{f^{-1}(B)} : f^{-1}(B) \to B$ is a homeomorphism. Then the lifted map $F: N^* \to N^*$ on the universal cover is also of geometric degree one, so it is algebraically of degree one, even when we compute with $\Lambda = Z[\pi_1(N)]$-coefficients. Letting $f^*\Lambda$ denote the pull-back bundle induced by f, Lemma 2.2 of [27] attests that $f_* : H_q(N; f^*\Lambda) \to H_q(N; \Lambda)$ is a split epimorphism, for all q. The groups above are isomorphic, for $f_* : \pi_1(N) \to \pi_1(N)$ is just a conjugation. The Noetherian hypothesis ensures that the split epimorphisms f_* on H_q are isomorphisms ($H_q(N; f^*\Lambda)$ is finitely generated, because submodules of finitely generated free Λ-modules are finitely generated in case Λ is Noetherian; moreover, $H_q(N; f^*\Lambda) \approx H_q(N; \Lambda) + \ker(f_*)$). Hence, f is a homotopy equivalence [28, Theorem 1] (cf. also [27, p. 22]).

Translating into the case at hand, we see that $r_g|_{g'} : g' \to g$ is a shape equivalence. This establishes the stability of the decomposition map. □

Recall that a group H is said to be *Hopfian* if every epimorphism $H \to H$ is an isomorphism.

Theorem 5.10. *Suppose G is a usc decomposition of an $(n+k)$-manifold M into compacta, all having the shape of some fixed orientable n-manifold N such that $\pi_1(N)$ is Hopfian and $Z[\pi_1(N)]$ is Noetherian. In addition, suppose* (1) $\dim[B = M/G] < \infty$ *and* (2) *the n-winding functions α_b defined on B are locally constant. Then B is a generalized k-manifold and the decomposition map $p: M \to B$ is an approximate fibration.*

Proof. Take $g \in G$. There is a shape retraction $r_g : U_g \to g$ defined on some neighborhood of g. By (2), g has a smaller neighborhood W_g such that $(r_g)_* : H_n(g') \to H_n(g)$ is an isomorphism for all $g' \in G$ in W_g. Consequently, r_g induces an epimorphism of

$$\pi_1(N) \approx \pi_1(g') \to \pi_1(g) \approx \pi_1(N),$$

which must be an isomorphism, since the group is Hopfian. This readily translates into Condition (ii) of Theorem 5.9, which result then yields the desired conclusion. □

The final results of this paper involve further applications of Theorem 3.1.

Lemma 5.11. *If M is an $(n+k)$-manifold, $n \geq k \geq 2$, and G is a usc decomposition of M into compacta having the shape of closed n-manifolds such that the reduced homology $H_i(g) \approx 0$ for $i \in \{0, 1, \ldots, k-1\}$ and $g \in G$, then the n-winding functions defined on $B = M/G$ are locally constant.*

Proof. By [12] in case $k=2$ or by the proof of Theorem 3.1 here in case $k \geq 3$, $H_k(B, B-\{b\}) \approx Z$ for each $b \in B$. Then the argument for the earlier Theorem 4.1 establishes the desired conclusion. □

Coupling Lemma 5.11 and Theorem 5.10, we obtain directly the following generalization of Theorem 4.1.

Theorem 5.12. *Suppose G is a usc decomposition of an $(n+k)$-manifold M, $n \geq k \geq 2$, into compacta, all having the shape of some fixed n-manifold N such that $\pi_1(N)$ is Hopfian, $Z[\pi_1(N)]$ is Noetherian and the reduced homology $H_i(N) \approx 0$ for $i \in \{0, 1, \ldots, k-1\}$. In addition, suppose $\dim[B = M/G] < \infty$. Then the decomposition map $p: M \to B$ is an approximate fibration.*

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